A New Conjecture On the primes

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Abstract. *In this article we present a new conjecture for consecutive primes that if* p_n *denoting the nth prime, then* $p_n^{\left(\frac{p_{n+1}}{p_n}\right)^n} \leq n^{p_n}$ *for every integer* $n > 4$ *. Initial we verification this conjecture for a finite sequence of primes. Afterward we prove that this new conjecture is stronger than other strong Nicholson's conjecture* $\left(\frac{p_{n+1}}{p_n}\right)^n \leq n \log n$ *for* $n > 4$ *. Also we compare the new conjecture with Nicholson's and Firoozbakht's and Cramer's and Granville's conjecture and in last we conclusion that the new conjecture is stronger than other conjectures.*

Key words: Primes, Cramer's conjecture, Firoozbakht's conjecture, Granville's conjecture, New conjecture, Nicholson's conjecture.

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1 Introduction

Given tow following strong conjectures about primes:

i) Firoozbakht's conjecture (1982): [7] *if* p_n *denoting the nth prime, then*

$$p_{n+1} \leq p_n^{\frac{n+1}{n}}$$

In fact Firoozbakht's conjecture statement that the sequence $\left\{p_n^{\frac{1}{n}}\right\}_{n\geq 1}$ is strictly decreasing. This conjecture is true for all primes up to 4×10^{18} [6]. Firoozbakht's conjecture is stronger than Cramer's conjecture $p_{n+1} - p_n \leq O((\log p_n)^2)$, [3],[6]. For more than details about Firoozbakht's conjecture, see [2], [5], [10].

ii) Nicholson's conjecture (2013): [11] *if* p_n *denoting the nth prime, then*

$$\left(\frac{p_{n+1}}{p_n}\right)^n \leq n \log n \text{ , } \forall n > 4$$

The Nicholson's conjecture is stronger than Firoozbakht's conjecture [8, Theorem 4.4].

In this article we will present a new very strong conjecture that implies the Nicholson's and Firoozbakht's conjecture and some other unsolved problems in this subject. In the next section we present the new conjecture.

2 New conjecture and its computational verification

In this section, we present the following conjecture about consecutive primes.

Conjecture. *If p_n denote the nth prime, then for every $n > 4$, we have*

$$p_n^{\left(\frac{p_{n+1}}{p_n}\right)^n} \leq n^{p_n} \tag{1}$$

If we take logarithm base p_n of inequality (1), we obtain the following inequality that similar to Nicholson's conjecture. So we have

$$\left(\frac{p_{n+1}}{p_n}\right)^n \leq p_n \log_{p_n} n \tag{2}$$

Also inequality (2) is equivalent to following inequality that similar to Firoozbakht's conjecture.

$$p_{n+1} \leq p_n^{\frac{n+1}{n}} \left(\log_{p_n} n\right)^{\frac{1}{n}} \tag{3}$$

Therefore is immediacy that the new conjecture is stronger than Firoozbakht's conjecture, because $\log_{p_n} n < 1$.

This conjecture is directly verify by computer computation, for sequence of first 10^4 primes. For this verify see table 1.

n	p_n	$\left(\frac{p_{n+1}}{p_n}\right)^n$	$\log_{p_n} n$	$p_n \log_{p_n} n$	$\left(\frac{p_{n+1}}{p_n}\right)^n \leq p_n \log_{p_n} n$
1	2	1.50	0.00	0.00	False
2	3	2.77	0.63	1.89	False
3	5	2.74	0.68	3.4	Ok
4	7	6.09	0.71	4.97	False
5	11	2.30	0.67	7.37	Ok
6	13	5.00	0.74	9.62	Ok
7	17	2.17	0.68	11.58	Ok
...	…	…	…	…	…
9996	104707	1.46	0.79	82718.53	Ok
9997	104711	1.77	0.79	82721.69	Ok
9998	104717	1.77	0.79	82726.43	Ok
9999	104723	1.77	0.79	82731.17	Ok
10000	104729	…	…	…	…

Table 1: Verification of inequality (2) of the new conjecture.

Corollary. *The new conjecture is true for all primes between 7 and 10^4th prime.*

3 New conjecture and Nicholson's conjecture

By [8] we know that the Nicholson's conjecture that presented by Nicholson in 2013 [11], is stronger than other strong Firoozbakht's conjecture. Now, in this section we want to prove that the new conjecture, is stronger than Nicholson's conjecture.

Lemma 1. *If p_n denote the nth prime, then for every integer $n \geq 4$, we have*

$$\frac{p_n}{\log p_n} \leq n \tag{4}$$

Proof. For proof of this lemma we use of corollary 1 from [1] or you can see inequality (4.1) from [4].

Theorem. *The new conjecture is stronger than Nicholson's conjecture.*

Proof. Let inequality (1) is true. So for $n>4$, we have

$$p_n{}^{\left(\frac{p_{n+1}}{p_n}\right)^n} \leq n^{p_n}$$

Now we take natural logarithm of above inequality and we obtain

$$\left(\frac{p_{n+1}}{p_n}\right)^n \log p_n \leq p_n \log n$$

Hence if above inequality division by $\log p_n$, we obtain

$$\left(\frac{p_{n+1}}{p_n}\right)^n \leq \frac{p_n \log n}{\log p_n}$$

By the previous lemma we know $\frac{p_n}{\log p_n} \leq n$ for $n \geq 4$. So, within the above inequality $\frac{p_n \log n}{\log p_n} \leq n \log n$. Therefore we have

$$\left(\frac{p_{n+1}}{p_n}\right)^n \leq \frac{p_n \log n}{\log p_n} \leq n \log n, \forall n > 4$$

And we know Nicholson's conjecture asserts that $\left(\frac{p_{n+1}}{p_n}\right)^n \leq n \log n$ for $n > 4$. And proof is complete.

Hence, according to above theorem, the conjectural bound for primes or gap between consecutive primes by the new conjecture is sharper than other conjectural bounds in this subject. So far this new conjecture is strongest conjecture in this subject. In the next section we will compare some conjectures related to the new conjecture.

4 Comparison

Several conjectures that are similar, they put in a common class. In relation to primes, we can statement that the new conjecture in this article and Cramer's, Firoozbakht's , Granville's [9] and Nicholson's conjectures are in a common class. But, which conjecture is stronger than other conjectures? By [5], [6] we know Firoozbakht's conjecture is stronger than Cramer's conjecture. Also recently the new chart available in [12], showed that the Firoozbakht's conjecture is stronger than Cramer's and Granville's conjectures.

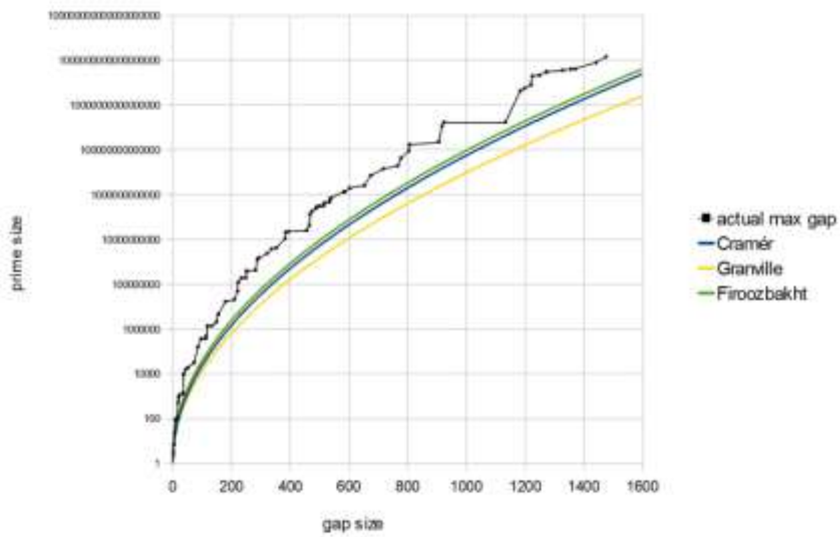

Figure 1. Firoozbakht's conjecture is stronger than Cramer's and Granville's conjecture.

Verify doing by maximal gaps between consecutive primes.

Chart available in [12].

Moreover by theorem (4.4) from [8] the Nicholson's conjecture is stronger than Firoozbakht's conjecture. On the other hand in the previous section we proved that the new conjecture is stronger than Nicholson's conjecture. Now in this section we want to compare these conjectures by computer program **R** .

1) Logical compare

R> P =C(primes between 7 and 10000th prime)

R> n=5:9999

R> Firoozbakht= bounded for P_n+1 by Firoozbakht's conjecture = P^((n+1)/n)

R> Nicholson= bounded for P_n+1 by Nicholson's conjecture = P *(n*log(n))^(1/n)

R> New= bounded for P_n+1 by new conjecture = P*(P*log(n, P))^(1/n)

R> New > Nicholson

True

R> New > Firoozbakht

True

2) Compare the New conjecture with Firoozbakht's, Nicholson's, Cramer's and Granville's conjectures, by computer program **R** and using the their conjectural upper bound for gaps between consecutive primes.

Initial, consider the conjectural upper bound for gap between consecutive primes by these conjectures in the following table.

Conjecture	**Conjectural upper bound for $\boldsymbol{p_{n+1}-p_n}$**
Cramer	$O((\log p_n)^2)$
Firoozbakht	$p_n^{\frac{n+1}{n}} - p_n, \forall n \geq 1$
Granville	$R \times O((\log p_n)^2), \quad R \sim 1.23$
New conjecture	$p_n\left(p_n \log_{p_n} n\right)^{\frac{1}{n}} - p_n, \forall n > 4$
Nicholson	$p_n(n \log n)^{\frac{1}{n}} - p_n, \forall n > 4$

Table 2. Conjectural upper bound for *n*th prime gaps.

Now, by plotting the charts of above conjectural upper bounds, and also chart of actual real data of gap between consecutive primes, we have

```
R> p=scan()
```

1: $prime\ p_n$, $5 \leq n \leq 10000$

```
R> n=5:10000

R> t=(n+1)/n

R> s=1/n

R> actual real data=seq(from=2, to= 72, length=9996)

R> plot(n,p*(p*log(n,p))^s-p,type="line",col="red",xlab=''n'',ylab=''upper bound for prime gaps by conjectures'')

R> lines(n, p*(n*log(n))^s-p, col="black")

R> lines(n, p^t-p, col="green")

R> lines(n,(log(p))^2,col="violet")

R> lines(n,1.23*(log(p))^2,col="yellow")

R> lines(n, actual real data, col=''brown'')
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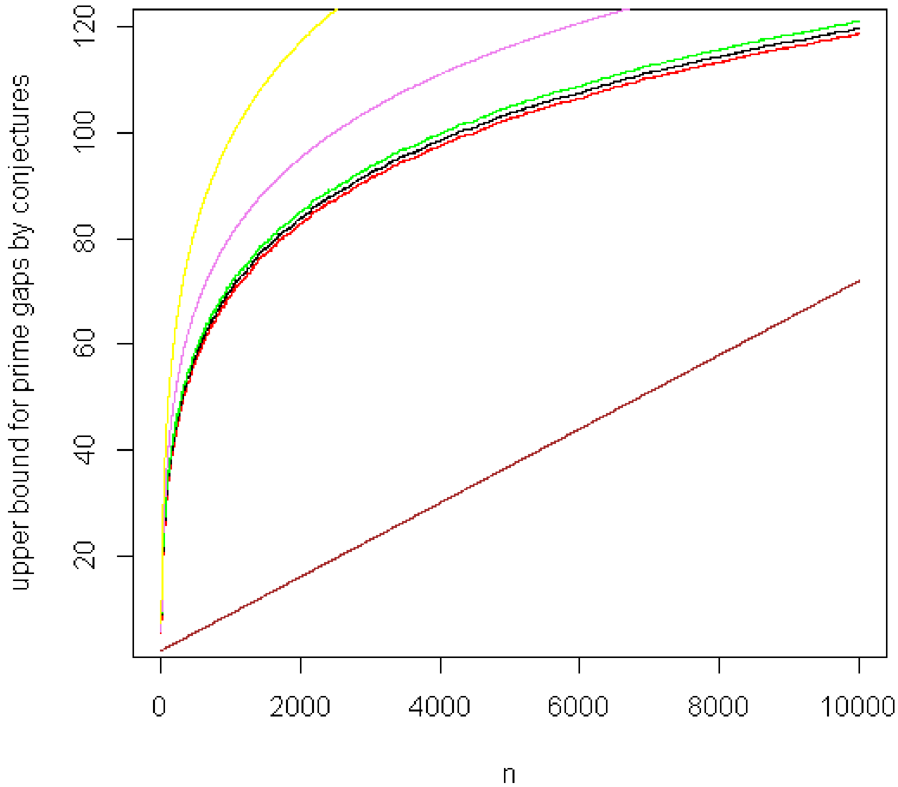

Figure 2. Graph of upper bound for gap between consecutive primes by *Granville's* , *Cramer's*, *Firoozbakht's*, *Nicholson's* and *New* conjecture and with the actual real data of prime gaps for first 10000 primes.

As shown in figure 2, the crump of upper bound for gap between consecutive primes by the new conjecture is lower than other conjectures and upper than actual real data line. Innuendo, conjectural values of upper bound for prime gaps by the new conjecture are between actual values and other conjectural values. Therefore the new conjecture is sharper than other conjectures.

Therewith, by theorem from section (3) in this article and by theorem (4.4) from [8], we conclusion that

$$\text{New conjecture} \Rightarrow \text{Nicholson}'\text{sconjecture} \Rightarrow \text{Firoozbakht}'\text{s conjecture}$$

And also by [6], [9] and figure 1, we know

$$\text{Firoozbakht}'\text{s conjecture} \Rightarrow \text{Cramer}'\text{s conjecture} \Rightarrow \text{Granville's conjecture}$$

Hence

$$\text{New conjecture} \Rightarrow \text{Nicholson}'\text{s conjecture} \Rightarrow \text{Firoozbakht}'\text{s conjecture} \Rightarrow \text{Cramer}'\text{s conjecture} \Rightarrow \text{Granville}'\text{s conjecture}$$

Therefore the following figure is a right conclusion of above results

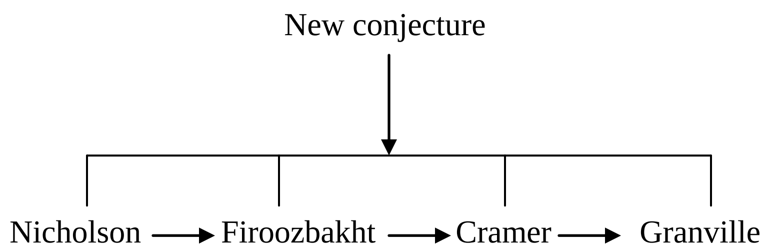

Figure 3. Preference for prove of conjectures.

By above figure if conjectural bound (1) is true, then all other conjectures are true. In fact according to this figure is best that initial new conjecture be proves and otherwise in pursuant of path by the symbol($\longrightarrow$) the other conjectures be proves.

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