Five integers which sum in pairs to squares

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I am most grateful to the *Mathematical Gazette* for publishing, in the March 1977 issue, my letter about the problem of finding five positive integers such that the sum of any pair is a perfect square. The response far exceeded my expectations and showed that the problem aroused great interest.

Many readers found solutions in which two or more of the integers are equal, but only a few managed to solve the more difficult problem of finding five integers which are both positive and distinct. This note describes their results, some found on computers but most by analytical methods. It also mentions briefly the work of Dr J. Lagrange of the University of Rheims and Dr J. L. Nicolas of the University of Limoges, which was not known to me at the time of my letter. I am indebted to Professor R. Guy of the University of Calgary for putting me in touch with Dr Lagrange.

The first complete solution to reach me came from Mr C. A. Glasbey, a student at Cambridge. He wrote a computer program which, in five minutes, calculated 4000 sets of four positive integers such that the sum of any pair is a square. At this point it identified the first set of five such integers, in such a way that the sum of the smallest and next smallest was a minimum. His five integers were

$$7442, \quad 28\,658, \quad 148\,583, \quad 177\,458, \quad 763\,442,$$

and the method by which he found them is briefly summarised in Annex A. Dr Lagrange states that these five numbers were first found in 1971 by Dr Nicolas (but of course unknown to Mr Glasbey), by a computer program which minimised the sum of all five of the numbers. Dr T. Jackson of the University of York also found some particular solutions using a computer.

However, the many readers of the *Gazette* who do not have access to a computer will be interested in the analytical solutions. The five numbers have to satisfy ten equations of the form

$$n_i + n_j = s_{ij}^2,$$

where $i, j = 1, 2, 3, 4, 5$ and $i \neq j$.

The analytical solvers fell into three groups. The first found parametric solutions of nine of the ten equations, and then solved the last. The second group found parametric solutions of eight equations, and then solved the last two. Finally, in a unique approach, Dr S. W. Dolan of Tunbridge Wells found a parametric solution of seven of the equations and then succeeded in reducing the last three to a single linear equation.

In the first group, Dr Lagrange found in 1971 a perfectly general parametric solution of nine of the ten equations. He then showed that the tenth equation reduces to the problem of finding a new rational point (x, y) on a quartic curve of the form

$$ax^4 + bx^3 + cx^2 + dx + e = y^2,$$

where one or more rational points are already known. This is a classical problem, which can be solved by methods due to Fermat and Euler.

Rear Admiral Sir Charles Darlington found a particular numerical solution of this kind. The largest of his five numbers had 27 digits.

In the second group of solutions there are two equations to solve. These reduce to the problem of finding rational numbers x, y, z to satisfy equations of the form:

$$ax^2 + bx + c = y^2, \quad dx^2 + ex + f = z^2.$$

This is another classical problem, known as the "double equation", and is closely related to that of the point on a quartic curve. My own solution, found in 1974, was of this type and led to five positive integers of which the largest had 12 digits. Dr G. A. Garreau of Streatham Hill found a similar method.

However, all these methods were exceeded in simplicity by that of Dr Dolan, which produced five positive integers of which the largest had 10 digits. His solution is summarised in Annex B.

In Annex C, I have reproduced (with his permission) a further parametric solution by Dr Lagrange, which leads to *six* integers (though not all positive) such that the sum of any pair is a square. The full derivation of this remarkable result is too lengthy to be given here, but it involves the combination of formulae for two magic squares, each with nine elements.

Finally, to complete the history of the problem, I should mention a paper by Mr T. Baker of Vauxhall, which appeared in *The gentleman's diary* in 1839. Although this did not give a numerical solution, it showed that the problem was possible in principle.

As Dr Lagrange was not a competitor, I have divided the prize of £25 between Mr Glasbey and Dr Dolan, who found the smallest and the simplest solutions, respectively. I should also like to thank all the other readers of the *Mathematical Gazette* who wrote to me, and I am very glad that they found the problem so interesting.

ANNEX A. *Mr Glasbey's computer program*

This note gives only a brief outline of the program, omitting various refinements. The calculation is designed to identify the positive integer solutions in a systematic order.

First consider three positive integers n_1, n_2, n_3 such that $n_1+n_2=a^2$, $n_1+n_3=b^2$, $n_2+n_3=c^2$. Without loss of generality, suppose that $n_1<n_2<n_3$ and hence $a<b<c$. Now $n_1=\frac{1}{2}(a^2+b^2-c^2)$, so $c^2<a^2+b^2$. Also $c\geqslant b+1$, so $(b+1)^2<a^2+b^2$, that is $2b<a^2-1$. These inequalities show that, for any given a, only a finite number of values of b and c are possible. For each successive $a=1, 2, 3, \ldots$, the computer identifies each possible value of b and c and each corresponding set n_1, n_2, n_3.

Next, for a given set n_1, n_2, n_3 we can search for all possible integers n_4 such that $n_4>n_3$ and $n_1+n_4=d^2$, $n_2+n_4=(d+e)^2$, $n_3+n_4=(d+f)^2$. It is easily seen that $d=(n_2-n_1-e^2)/2e$, so that $e^2<n_2-n_1$. Thus only a finite number of values of e, and hence of d, are possible. For each such d, it is evident that $n_4=d^2-n_1$ is determined, and it is then easy to test whether n_3+n_4 is square.

Next, the computer searches for a further integer n_5 which is greater than n_4 and is such that n_1+n_5, n_2+n_5, and n_3+n_5 are squares. Evidently n_5 must satisfy the same conditions as n_4, so only a finite number of values are possible. Finally, the computer tests whether n_4+n_5 is square.

Mr Glasbey's calculation showed that there is no solution for any value of $a<190$. When $a=190$ there is the solution:

$$n_1=7442,\quad n_2=28\,658,\quad n_3=148\,583,\quad n_4=177\,558,\quad n_5=763\,442.$$

ANNEX B. *Solution by Dr Dolan*

The five numbers n_1, n_2, n_3, n_4, n_5 have to satisfy the ten equations:

$$n_i+n_j=s_{ij}^2 \quad (i\neq j).$$

Plainly it is sufficient to find a solution in rational numbers, for these can always be converted to integers by multiplying all the ns by a suitable square.

If we take a, b, c, d, e as any rational numbers, and put:

$$\left.\begin{aligned} n_1&=2a^2+2bcde,\\ n_2&=2a^2-2bcde,\\ n_3&=b^2d^2e^2+c^2-2a^2,\\ n_4&=b^2c^2e^2+d^2-2a^2,\\ n_5&=b^2c^2d^2+e^2-2a^2, \end{aligned}\right\} \qquad (1)$$

then seven of the ten equations are satisfied, because n_1+n_2, n_1+n_3, n_1+n_4, n_1+n_5, n_2+n_3, n_2+n_4 and n_2+n_5 are automatically squares. It then remains to choose the parameters a, b, c, d, e so that $n_3 + n_4$, $n_3 + n_5$ and $n_4 + n_5$ are also squares.

Now let $n_3 + n_4 = s^2$. This can be written as

$$(b^2e^2 + 1)(c^2 + d^2) - (2a)^2 = s^2$$

and hence as

$$(2a)^2 + s^2 = (bce + d)^2 + (bde - c)^2,$$

which will be satisfied if

$$2a = x_1(bce + d) + x_2(bde - c), \tag{2}$$

where x_1 and x_2 are any rational numbers such that $x_1^2 + x_2^2 = 1$.

Similarly we find that $n_3 + n_5$ and $n_4 + n_5$ will be squares if

$$2a = x_3(bcd + e) + x_4(bde - c), \tag{3}$$

$$2a = x_5(bcd + e) + x_6(bce - d), \tag{4}$$

where $x_3^2 + x_4^2 = x_5^2 + x_6^2 = 1$.

Let us now choose the xs as any rational numbers which satisfy

$$x_1^2 + x_2^2 = x_3^2 + x_4^2 = x_5^2 + x_6^2 = 1, \tag{5}$$

and then regard (2), (3) and (4) as linear equations in a and b. These equations will be consistent if

$$\begin{vmatrix} -2 & cex_1 + dex_2 & dx_1 - cx_2 \\ -2 & cdx_3 + dex_4 & ex_3 - cx_4 \\ -2 & cdx_5 + cex_6 & ex_5 - dx_6 \end{vmatrix} = 0. \tag{6}$$

If this determinant is expanded and expressed as a quadratic in d, it is immediately seen that the coefficient of d^2 will vanish if c and e are chosen so that

$$\left.\begin{aligned} c &= x_1x_4 + x_4x_6 - x_2x_6, \\ e &= x_1x_5 - x_1x_3 - x_3x_6. \end{aligned}\right\} \quad (7$$

With these values, (6) reduces to a linear equation for d.

Thus Dr Dolan's solution proceeds as follows. We choose any xs which satisfy (5); we then find c and e from (7); determine d from (6); solve (2), (3) and (4) to obtain a and b; and finally substitute in (1) to obtain the ns.

In this way, taking the xs as combinations of $\pm 3/5$, $\pm 4/5$, Dr Dolan found the following five ns which solve the problem:

30 823 058, 63 849 842, 150 187 058, 352 514 183, 1 727 301 842.

ANNEX C. *Dr Lagrange's parametric solution*

By using formulae for magic squares, Dr Lagrange expressed six rational numbers in terms of a rational parameter x as follows:

$$\begin{aligned}
n_1 &= 2x^8 - 24x^7 + 100x^6 - 24x^5 - 334x^4 + 48x^3 + 400x^2 + 192x + 32,\\
n_2 &= 2x^8 + 24x^7 + 100x^6 + 24x^5 - 334x^4 - 48x^3 + 400x^2 - 192x + 32,\\
n_3 &= 2x^8 - 40x^6 + 258x^4 - 160x^2 + 32,\\
n_4 &= 2x^8 + 8x^7 - 60x^6 - 88x^5 + 466x^4 - 176x^3 - 240x^2 + 64x + 32,\\
n_5 &= 2x^8 - 8x^7 - 60x^6 + 88x^5 + 466x^4 + 176x^3 - 240x^2 - 64x + 32,\\
n_6 &= -x^8 + 50x^6 - 225x^4 + 200x^2 - 16.
\end{aligned}$$

These six numbers have the property that the sum of any pair is a square, *except* for the pair $n_1 + n_2$. Thus the five numbers n_1, n_3, n_4, n_5, n_6, are such that the sum of any pair is a square, and the same is true of the five numbers n_2, n_3, n_4, n_5, n_6. All these numbers will be positive if the parameter x is taken in the neighbourhood of 1.

Furthermore, Dr Lagrange showed that $n_1 + n_2$ is also a square when $x = 14/5$. There are then *six* numbers (though they are not all positive) such that the sum of any pair is a square. Converted to integers, the six numbers are

−15 863 902, 17 798 783, 21 126 338, 49 064 546,
82 221 218, 447 422 978.

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