International Journal of Number Theory
(2015)

DOI: 10.1142/S1793042115500281

Sextuples of integers whose sums in pairs are squares

Ajai Choudhry
13/4 A Clay Square
Lucknow 226001, India
ajaic203@yahoo.com

Received 1 April 2014
Accepted 18 June 2014
Published 23 July 2014

This paper is concerned with the diophantine problem of finding six integers such that the sum of any two of them is a perfect square. Till now, only one numerical example of such a sextuple has been published. In this paper, we obtain infinitely many examples of sextuples of integers such that the sum of any two of them is a perfect square. These examples include sextuples which have three or four or five distinct integers as well as sextuples in which all the integers are distinct.

Keywords: Sextuples of integers; sums in pairs are squares.

Mathematics Subject Classification 2010: 11D09

1. Introduction

Erdős and Moser have posed the following diophantine problem [4, p. 268]:

"Are there for every n, n distinct integers such that the sum of any two is a perfect square?"

When $n = 3$ or 4, it is fairly straightforward to find formulae that yield integers whose sums in pairs are squares (see [4, p. 268]). Further, when $n = 5$, numerical examples of such integers have been given by several authors (see [3, p. 455; 7]). A method of obtaining quintuples of integers, in parametric terms, with the desired property is given in [5] but the resulting solution is extremely cumbersome and not given explicitly by the author. Choudhry [1] has recently obtained quintuples of distinct polynomials of degree six such that the sum of any two of them is a perfect square. These naturally yield infinitely many quintuples of integers whose pairwise sums are squares. When $n = 6$, till now only one numerical example of such integers is known [6, p. 94].

In this paper we find sextuples of integers whose sums in pairs are perfect squares. The problem of finding such sextuples remains a nontrivial problem even when it is not stipulated that the six integers should be all distinct. Therefore, apart from obtaining such sextuples of distinct integers, we also obtain such sextuples which include repeated integers.

We first note that any sextuple of rational numbers whose pairwise sums are perfect squares leads, on multiplying through by a suitable perfect square, to a sextuple of integers with the same property. It therefore suffices to obtain sextuples of rational numbers with the desired property.

2. A Preliminary Lemma

It would be recalled that a square matrix is said to be a semi-magic matrix if the sum of the entries in each row and each column is the same. This common sum is referred to as the semi-magic sum. The following lemma relates the existence of sextuples whose pairwise sums are squares to three 3×3 semi-magic matrices which satisfy certain conditions.

Lemma 1. *There exist six rational numbers* $n_i, i = 1, 2, \dots, 6$, *whose sums in pairs are squares if and only if there exist three* 3×3 *semi-magic matrices* A, B, C, *with squared entries, namely,*

$$A = \begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 \\ a_{21}^2 & a_{22}^2 & a_{23}^2 \\ a_{31}^2 & a_{32}^2 & a_{33}^2 \end{bmatrix}, \quad B = \begin{bmatrix} b_{11}^2 & b_{12}^2 & b_{13}^2 \\ b_{21}^2 & b_{22}^2 & b_{23}^2 \\ b_{31}^2 & b_{32}^2 & b_{33}^2 \end{bmatrix}, \quad C = \begin{bmatrix} c_{11}^2 & c_{12}^2 & c_{13}^2 \\ c_{21}^2 & c_{22}^2 & c_{23}^2 \\ c_{31}^2 & c_{32}^2 & c_{33}^2 \end{bmatrix},$$

such that the following relations are satisfied:

$$a_{1j} = b_{1j} = c_{1j}, \quad j = 1, 2, 3, \tag{2.1}$$

$$a_{21} = b_{21}, \quad a_{31} = b_{31}, \tag{2.2}$$

$$a_{23} = c_{23}, \quad a_{33} = c_{33}, \tag{2.3}$$

$$b_{22} = c_{32}, \quad b_{32} = c_{22}. \tag{2.4}$$

Further, when such matrices exist, the rational numbers $n_i, i = 1, 2, \dots, 6$, *defined by*

$$\begin{aligned} n_1 &= (a_{11}^2 + a_{22}^2 - b_{22}^2)/2, \\ n_2 &= (a_{11}^2 - a_{22}^2 + b_{22}^2)/2, \\ n_3 &= (-a_{11}^2 - 2a_{13}^2 + 2a_{21}^2 + a_{22}^2 + b_{22}^2)/2, \\ n_4 &= (a_{11}^2 + 2a_{12}^2 + 2a_{13}^2 - 2a_{21}^2 - a_{22}^2 - b_{22}^2)/2, \\ n_5 &= (a_{11}^2 + 2a_{13}^2 - a_{22}^2 - b_{22}^2)/2, \\ n_6 &= (-a_{11}^2 + a_{22}^2 + b_{22}^2)/2, \end{aligned} \tag{2.5}$$

are such that their sums in pairs are squares.

Proof. If there exist six integers $n_i, i = 1, 2, \ldots, 6$, whose sums in pairs are squares, then it is readily seen that the three matrices,

$$A = \begin{bmatrix} n_1 + n_2 & n_3 + n_4 & n_5 + n_6 \\ n_3 + n_5 & n_1 + n_6 & n_2 + n_4 \\ n_4 + n_6 & n_2 + n_5 & n_1 + n_3 \end{bmatrix},$$

$$B = \begin{bmatrix} n_1 + n_2 & n_3 + n_4 & n_5 + n_6 \\ n_3 + n_5 & n_2 + n_6 & n_1 + n_4 \\ n_4 + n_6 & n_1 + n_5 & n_2 + n_3 \end{bmatrix}, \tag{2.6}$$

$$C = \begin{bmatrix} n_1 + n_2 & n_3 + n_4 & n_5 + n_6 \\ n_3 + n_6 & n_1 + n_5 & n_2 + n_4 \\ n_4 + n_5 & n_2 + n_6 & n_1 + n_3 \end{bmatrix},$$

are such that all the entries of these matrices are perfect squares, the sum of each row and each column of these matrices is $n_1 + n_2 + n_3 + n_4 + n_5 + n_6$ and thus all of them are semi-magic matrices, and the relations (2.1), (2.2), (2.3), (2.4), are all satisfied.

Conversely, if there exist three semi-magic matrices A, B, C, with squared entries satisfying the conditions (2.1), (2.2), (2.3), (2.4), they have the same semi-magic sum s. We will now show that the rational numbers n_i defined by (2.5) are such that the sum of any two of them is a perfect square.

Since A is a semi-magic matrix, we have

$$\begin{aligned} &n_1 + n_2 = a_{11}^2, \quad n_3 + n_4 = a_{12}^2, \quad n_5 + n_6 = a_{13}^2, \\ &n_3 + n_5 = a_{21}^2, \quad n_1 + n_6 = a_{22}^2, \\ &n_2 + n_4 = a_{11}^2 + a_{12}^2 + a_{13}^2 - a_{21}^2 - a_{22}^2 = a_{23}^2, \\ &n_4 + n_6 = a_{12}^2 + a_{13}^2 - a_{21}^2 = a_{31}^2, \\ &n_2 + n_5 = a_{11}^2 + a_{13}^2 - a_{22}^2 = a_{32}^2, \\ &n_1 + n_3 = a_{21}^2 + a_{22}^2 - a_{13}^2 = a_{33}^2. \end{aligned}$$

Next, using the relations (2.1), (2.2) and the fact that B is a semi-magic matrix, we have

$$\begin{aligned} n_2 + n_6 &= b_{22}^2, \\ n_1 + n_4 &= a_{11}^2 + a_{12}^2 + a_{13}^2 - a_{21}^2 - b_{22}^2 = s - b_{21}^2 - b_{22}^2 = b_{23}^2, \\ n_1 + n_5 &= a_{11}^2 + a_{13}^2 - b_{22}^2 = b_{11}^2 + b_{13}^2 - b_{22}^2 = b_{32}^2, \\ n_2 + n_3 &= -a_{13}^2 + a_{21}^2 + b_{22}^2 = -b_{13}^2 + b_{21}^2 + b_{22}^2 = b_{33}^2. \end{aligned}$$

Finally using the relations (2.1), (2.3), (2.4) and the fact that A and C are both semi-magic, we have

$$\begin{aligned} n_3 + n_6 &= -a_{11}^2 - a_{13}^2 + a_{21}^2 + a_{22}^2 + b_{22}^2 \\ &= a_{12}^2 - a_{23}^2 + c_{32}^2 = c_{12}^2 - c_{23}^2 + c_{32}^2 = c_{21}^2, \\ n_4 + n_5 &= a_{11}^2 + a_{12}^2 + 2a_{13}^2 - a_{21}^2 - a_{22}^2 - b_{22}^2 \\ &= a_{13}^2 + a_{23}^2 - c_{32}^2 = c_{13}^2 + c_{23}^2 - c_{32}^2 = c_{31}^2. \end{aligned}$$

Thus all fifteen pairwise sums of the six numbers n_i are perfect squares, and the proof is complete. □

We note here that all the conditions stated in Lemma 1 are not independent. In fact, when there exist semi-magic matrices satisfying condition (2.1), the rows and columns of the three semi-magic matrices add up to the same common sum and therefore, either of the two conditions mentioned in (2.2) or (2.3) or (2.4) implies the other.

3. Sextuples Whose Pairwise Sums are Squares

It is a trivial exercise to find all sextuples of rational numbers only two of which are distinct and whose pairwise sums are squares. We give below, in Secs. 3.1, 3.2 and 3.3 respectively, sextuples of rational numbers in which exactly three, four or five of the numbers are distinct, and whose pairwise sums are squares. Finally in Sec. 3.4 we find sextuples of distinct numbers all of whose 15 pairwise sums are squares.

3.1. *Sextuples in which three numbers are distinct*

We note that any repeated number(s) in the sextuple must necessarily be of the type $k^2/2$, and hence the sextuple must be of one of the following types:

(i) $k^2/2, k^2/2, k^2/2, k^2/2, x_1^2 - k^2/2, x_2^2 - k^2/2$;
(ii) $k^2/2, k^2/2, k^2/2, m^2/2, m^2/2, x^2 - k^2/2$;
(iii) $x_1^2/2, x_1^2/2, x_2^2/2, x_2^2/2, x_3^2/2, x_3^2/2$.

Choudhry [1, Sec. 3.1] has obtained all quintuples of rational numbers of the type $k^2/2, k^2/2, k^2/2, x_1^2 - k^2/2, x_2^2 - k^2/2$ and $k^2/2, k^2/2, m^2/2, m^2/2, x^2 - k^2/2$ such that all pairwise sums of these quintuples are perfect squares. By simply appending $k^2/2$ to these quintuples, we obtain the first two types of sextuples mentioned above. We will now obtain sextuples of the third type.

While it is possible to apply Lemma 1 to obtain the desired sextuples, the following direct approach is simpler and straightforward. All pairwise sums of the sextuple $x_1^2/2, x_1^2/2, x_2^2/2, x_2^2/2, x_3^2/2, x_3^2/2$ will be perfect squares if the following three diophantine equations are satisfied:

$$x_1^2 + x_2^2 = 2u^2, \tag{3.1}$$

$$x_2^2 + x_3^2 = 2v^2, \tag{3.2}$$

$$x_3^2 + x_1^2 = 2w^2. \tag{3.3}$$

Equation (3.1) may be written as $x_1^2 - u^2 = u^2 - x_2^2$, and may therefore be replaced by the following two equations:

$$x_1 - u = s(u - x_2), \quad s(x_1 + u) = u + x_2, \tag{3.4}$$

where s is some rational number. Similarly Eq. (3.2) may be replaced by the following two equations:

$$x_2 - v = t(v - x_3), \quad t(x_2 + v) = v + x_3, \tag{3.5}$$

where t is some rational number. Now (3.4) and (3.5) may be considered as four linear equations in the five variables x_1, x_2, x_3, u, v, and are hence readily solved. We thus get

$$\begin{aligned} x_1 &= (s^2 - 2s - 1)(t^2 - 2t - 1), \quad x_2 = -(s^2 + 2s - 1)(t^2 - 2t - 1), \\ x_3 &= (s^2 + 2s - 1)(t^2 + 2t - 1), \quad u = -(s^2 + 1)(t^2 - 2t - 1), \\ v &= (t^2 + 1)(s^2 + 2s - 1), \end{aligned} \tag{3.6}$$

and on substituting these values of x_1, x_3 in (3.3), we get the condition,

$$\begin{aligned} (t^2 + 1)^2 s^4 + 16t(t^2 - 1)s^3 + 2(t^2 + 1)^2 s^2 \\ - 16t(t^2 - 1)s + (t^2 + 1)^2 = w^2. \end{aligned} \tag{3.7}$$

Now the left-hand side of (3.7) is a quartic function of s and it can be made a perfect square using the classical method given by Fermat (quoted by Dickson [3, p. 639]). We thus find that $s = -(t^2 + 1)^2/\{2t(t^2 - 1)\}$ gives a solution of (3.7), and this leads to the following sextuples whose sums in pairs are squares:

$$\begin{aligned} n_1 = n_2 &= 2(t^2 - 2t - 1)^2(t^4 + 4t - 1)^2(t^4 + 4t^3 - 1)^2, \\ n_3 = n_4 &= 2(t^2 - 2t - 1)^2(t^4 - 4t - 1)^2(t^4 - 4t^3 - 1)^2, \\ n_5 = n_6 &= 2(t^2 + 2t - 1)^2(t^4 - 4t - 1)^2(t^4 - 4t^3 - 1)^2, \end{aligned} \tag{3.8}$$

where t is an arbitrary rational parameter. As a numerical example, taking $t = 2$, we get the sextuple

$$2337122, 2337122, 28322, 28322, 1387778, 1387778 \tag{3.9}$$

whose sums in pairs are squares.

We note that using the known solution of (3.7), we can, by repeatedly applying the method of Fermat, obtain infinitely many solutions of (3.7), and hence obtain infinitely many such sextuples in parametric terms such that their pairwise sums are squares.

3.2. *Sextuples in which four numbers are distinct*

Such sextuples can be of the following two types:

(i) $k^2/2, k^2/2, k^2/2, x_1^2 - k^2/2, x_2^2 - k^2/2, x_3^2 - k^2/2$;
(ii) $k^2/2, k^2/2, m^2/2, m^2/2, x_1^2 - k^2/2, x_2^2 - k^2/2$.

As before, sextuples of the first type are obtained by appending $k^2/2$ to quintuples of the type $k^2/2, k^2/2, x_1^2 - k^2/2, x_2^2 - k^2/2, x_3^2 - k^2/2$ given in [1, Sec. 3.2]. We will now apply Lemma 1 to obtain sextuples of the second type mentioned above.

Euler gave the following semi-magic matrix (as quoted by Dickson [3, p. 530]):

$$M = \begin{bmatrix} (p^2+q^2-r^2-s^2)^2 & (2qr+2ps)^2 & (2qs-2pr)^2 \\ (2qr-2ps)^2 & (p^2-q^2+r^2-s^2)^2 & (2pq+2rs)^2 \\ (2qs+2pr)^2 & (2rs-2pq)^2 & (p^2-q^2-r^2+s^2)^2 \end{bmatrix}. \quad (3.10)$$

We note that interchanging any two columns of a semi-magic matrix leads to another semi-magic matrix. We now interchange the first two columns of M to obtain the following semi-magic matrix:

$$A = \begin{bmatrix} (2qr+2ps)^2 & (p^2+q^2-r^2-s^2)^2 & (2qs-2pr)^2 \\ (p^2-q^2+r^2-s^2)^2 & (2qr-2ps)^2 & (2pq+2rs)^2 \\ (2rs-2pq)^2 & (2qs+2pr)^2 & (p^2-q^2-r^2+s^2)^2 \end{bmatrix}.$$

We take B and C to be semi-magic matrices identical to A. The three identical semi-magic matrices satisfy all the conditions of the lemma except condition (2.4). This condition is easily satisfied by taking $r = t(p+q), s = t(-p+q)$ where t is an arbitrary parameter, and now, using (2.5), we get the following sextuple whose sums in pairs are squares:

$$\begin{aligned} n_1 = n_2 &= 2t^2(p^2-2pq-q^2)^2, \\ n_3 &= -(2t^2-1)\{p^4-(8t^2+2)p^2q^2+q^4\}, \\ n_4 &= 2(2t^2-1)\{t^2p^4-(2t^2+2)p^2q^2+t^2q^4\}, \\ n_5 = n_6 &= 2t^2(p^2+2pq-q^2)^2, \end{aligned} \quad (3.11)$$

where p, q and t are arbitrary parameters. As a numerical example, taking $p = 5$, $q = 2, t = 1$, we get the following sextuple consisting of positive integers only:

$$2, 2, 359, 482, 3362, 3362. \quad (3.12)$$

We can get another parametric solution, though not so simple as above, by beginning with three semi-magic matrices identical to M, imposing the condition (2.4) (the other conditions are automatically satisfied), and applying Lemma 1.

3.3. *Sextuples in which five numbers are distinct*

We will now obtain, by applying Lemma 1, sextuples in which five numbers are distinct and whose pairwise sums are squares. We take both A and B as identical to the semi-magic matrix M given by (3.10). We choose the third matrix C as follows:

$$C = \begin{bmatrix} (p^2+q^2-r^2-s^2)^2 & (2qr+2ps)^2 & (2qs-2pr)^2 \\ c_{21}^2 & (2rs-2pq)^2 & (2pq+2rs)^2 \\ c_{31}^2 & (p^2-q^2+r^2-s^2)^2 & (p^2-q^2-r^2+s^2)^2 \end{bmatrix}$$

where we will choose p, q, r, s, c_{21} and c_{31} such that C becomes a semi-magic matrix. The three matrices A, B, C already satisfy the conditions (2.1), (2.2), (2.3) and (2.4) and so on applying Lemma 1, we will get the following sextuple whose sums in pairs are squares:

$$\begin{aligned} n_1 = n_2 &= p^4 + 2p^2q^2 - 2p^2r^2 - 2p^2s^2 + q^4 - 2q^2r^2 \\ &\quad - 2q^2s^2 + r^4 + 2r^2s^2 + s^4, \\ n_3 &= p^4 - 6p^2q^2 - 2p^2r^2 + 6p^2s^2 + q^4 + 6q^2r^2 \\ &\quad - 2q^2s^2 + r^4 - 6r^2s^2 + s^4, \\ n_4 &= -p^4 + 6p^2q^2 + 2p^2r^2 + 2p^2s^2 + 16pqrs \\ &\quad - q^4 + 2q^2r^2 + 2q^2s^2 - r^4 + 6r^2s^2 - s^4, \\ n_5 &= -p^4 + 6p^2q^2 + 2p^2r^2 + 2p^2s^2 - 16pqrs \\ &\quad - q^4 + 2q^2r^2 + 2q^2s^2 - r^4 + 6r^2s^2 - s^4, \\ n_6 &= p^4 - 6p^2q^2 + 6p^2r^2 - 2p^2s^2 + q^4 \\ &\quad - 2q^2r^2 + 6q^2s^2 + r^4 - 6r^2s^2 + s^4. \end{aligned} \tag{3.13}$$

We will thus obtain sextuples in which five of the integers are distinct.

To choose $p, q, r, s, c_{21}, c_{31}$, such that C becomes a semi-magic matrix, we note that the sum of each row and each column of C must be the same as the sum of the first row, that is, $(p^2+q^2+r^2+s^2)^2$ and it suffices to impose the following conditions by which the sums of the second and third rows of C add up to this common sum:

$$c_{21}^2 + (2rs-2pq)^2 + (2pq+2rs)^2 = (p^2+q^2+r^2+s^2)^2, \tag{3.14}$$

and

$$\begin{aligned} c_{31}^2 &+ (p^2-q^2+r^2-s^2)^2 + (p^2-q^2-r^2+s^2)^2 \\ &= (p^2+q^2+r^2+s^2)^2. \end{aligned} \tag{3.15}$$

Now (3.14) may be rewritten as

$$\begin{aligned} c_{21}^2 &+ (2rs-2pq)^2 \\ &= (p^2+q^2+r^2+s^2)^2 - (2pq+2rs)^2, \\ &= (2qs-2pr)^2 + (p^2-q^2-r^2+s^2)^2, \\ &= \{(2qs-2pr)(t^2-1) - 2(p^2-q^2-r^2+s^2)t\}^2/(t^2+1)^2, \\ &\quad + \{2(2qs-2pr)t + (p^2-q^2-r^2+s^2)(t^2-1)\}^2/(t^2+1)^2, \end{aligned} \tag{3.16}$$

where t is an arbitrary rational parameter, and so we will get

$$c_{21} = \{(2qs-2pr)(t^2-1) - 2(p^2-q^2-r^2+s^2)t\}/(t^2+1), \tag{3.17}$$

if we impose the following condition:

$$2rs - 2pq = \{2(2qs-2pr)t + (p^2-q^2-r^2+s^2)(t^2-1)\}/(t^2+1). \tag{3.18}$$

Now (3.18) may be rewritten equivalently as

$$\begin{aligned} \{(p+q+r+s)t + p - q - r + s\}\{(p+q-r-s)t \\ -p+q-r+s\} = 2(q+s)(q-s)(t-1)(t+1), \end{aligned} \tag{3.19}$$

and (3.19) may be replaced by the following two equations:

$$\begin{aligned} (p+q+r+s)t + p - q - r + s &= 2m(q+s)(t-1), \\ m((p+q-r-s)t - p + q - r + s) &= (q-s)(t+1), \end{aligned} \tag{3.20}$$

where m is some rational number. Now the two equations in (3.20) may be considered as linear equations in r and s, and are therefore readily solved. We thus obtain the following solution of (3.20):

$$\begin{aligned} r &= \{(2(t-1)^2m^2 - 2(t^2-1)m + (t+1)^2)p + 2(2mt-t-1) \\ &\quad \times (mt-m-t)q\}\{2(t^2-1)m^2 - 4tm - t^2 + 1\}^{-1}, \\ s &= \{2(t^2+1)mp - (2m^2-2m+1)(t^2-1)q\} \\ &\quad \times \{2(t^2-1)m^2 - 4tm - t^2 + 1\}^{-1}. \end{aligned} \tag{3.21}$$

With these values of r, s and with c_{21} defined by (3.17), Eq. (3.14) is satisfied while Eq. (3.15) reduces to the following condition:

$$\begin{aligned} c_{31}^2 = \phi_0(m,t)p^4 + \phi_1(m,t)p^3q + \phi_2(m,t)p^2q^2 \\ + \phi_3(m,t)pq^3 + \phi_4(m,t)q^4, \end{aligned} \tag{3.22}$$

where $\phi_j(m,t), j = 0, 1, \ldots, 4$, are rational functions of m and t that can be effectively computed but are cumbersome to write and are hence not given explicitly. For suitably chosen numerical values of m, t and q, Eq. (3.22) becomes a quartic equation in c_{31} and p that represents an elliptic curve of positive rank. As an example, when $m = 1, t = 2, q = 1$, Eq. (3.22) reduces to

$$c_{31}^2 = 16(625p^4 - 750p^3 + 975p^2 - 330p + 9)/625, \tag{3.23}$$

which is a quartic model of an elliptic curve. The birational transformation given by

$$\begin{aligned} p &= \frac{3\xi - 72}{5(6\xi + \eta - 17)}, \\ c_{31} &= \frac{12\xi^3 - 864\xi^2 - 840\xi - 2760\eta + 1416}{25(6\xi + \eta - 17)^2}, \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} \xi &= \frac{700p^2 - 660p + 75c_{31} + 36}{200p^2}, \\ \eta &= \frac{-4000p^3 + 7500p^2 - 2250pc_{31} - 3060p + 225c_{31} + 108}{1000p^3} \end{aligned} \tag{3.25}$$

transforms the quartic model (3.23) to the Weierstrass form,

$$\eta^2 + \xi\eta = \xi^3 - \xi^2 - 9\xi + 49. \tag{3.26}$$

This is an elliptic curve of rank 3 as may be readily confirmed from Cremona's well-known database on elliptic curves [2]. Computations related to the curve (3.26) can be readily performed using any software (e.g., APECS) for working with elliptic curves. The three generators of the group of rational points on the curve (3.26) are found to be $P_1 = (2, 5), P_2 = (-4, -1)$ and $P_3 = (-1, -7)$. Using the group law we can now compute infinitely many rational points on the elliptic curve (3.26), and working backwards, using (3.24) and (3.21), we can find infinitely many solutions of the diophantine equations (3.14) and (3.15), and eventually, using the relations (3.13), we obtain infinitely many sextuples whose sums in pairs are squares. While we may get sextuples including one negative integer, it is possible to get sextuples of positive integers only. As a numerical example, the point $4P_1$ on the curve (3.26) leads to the following sextuple which consists only of positive integers and which is such that the sum of any two of the integers is a perfect square:

$$\begin{array}{ccc} 3694388882, & 3694388882, & 60445225682, \\ 42248104082, & 102804712082, & 254645020559. \end{array} \tag{3.27}$$

3.4. *Sextuples in which all the six numbers are distinct*

To find sextuples of distinct numbers such that all pairwise sums are squares, we will first construct three semi-magic matrices A, B, C, satisfying the conditions of Lemma 1, and then obtain the sextuples using (2.5). In the semi-magic matrix M given by (3.10), we replace p, q, r, s by $pu - qv$, $pv + qu$, $su - rv$, $sv + ru$, respectively and then divide each entry of the resulting semi-magic matrix by $(u^2+v^2)^2$ to get the semi-magic matrix $A=(a_{ij})$ whose entries are given below:

$$\begin{aligned}
a_{11} &= (p^2+q^2-r^2-s^2)^2,\\
a_{12} &= 4(ps+qr)^2,\\
a_{13} &= 4(pr-qs)^2,\\
a_{21} &= 4\{(-ps+qr)u^2+2(pr+qs)uv+(ps-qr)v^2\}^2/k_1^2,\\
a_{22} &= \{(p^2-q^2+r^2-s^2)u^2-4(pq-rs)uv\\
&\quad -(p^2-q^2+r^2-s^2)v^2\}^2/k_1^2,\\
a_{23} &= 4\{(pq+rs)u^2+(p^2-q^2-r^2+s^2)uv-(pq+rs)v^2\}^2/k_1^2,\\
a_{31} &= 4\{(pr+qs)u^2+2(ps-qr)uv-(pr+qs)v^2\}^2/k_1^2,\\
a_{32} &= 4\{(pq-rs)u^2+(p^2-q^2+r^2-s^2)uv-(pq-rs)v^2\}^2/k_1^2,\\
a_{33} &= \{(p^2-q^2-r^2+s^2)u^2-4(pq+rs)uv\\
&\quad -(p^2-q^2-r^2+s^2)v^2\}^2/k_1^2,
\end{aligned} \tag{3.28}$$

where $k_1=u^2+v^2$.

In the semi-magic matrix A, we replace u by $(-ps+qr)u+(pr+qs)v$ and v by $(pr+qs)u-(-ps+qr)v$ to get another semi-magic matrix $B=(b_{ij})$ whose entries $b_{11},b_{12},b_{13},b_{21},b_{22}$ and b_{31} are given by

$$\begin{aligned}
b_{1j} &= a_{1j},\quad j=1,2,3,\quad b_{21}=a_{21},\quad b_{31}=a_{31},\\
b_{22} &= \{(p^4r^2-p^4s^2+2p^2q^2r^2-2p^2q^2s^2+p^2r^4+2p^2r^2s^2\\
&\quad +p^2s^4+q^4r^2-q^4s^2-q^2r^4-2q^2r^2s^2-q^2s^4)u^2\\
&\quad +(4p^4rs+8p^2q^2rs-4pqr^4-8pqr^2s^2-4pqs^4+4q^4rs)uv\\
&\quad +(-p^4r^2+p^4s^2-2p^2q^2r^2+2p^2q^2s^2-p^2r^4-2p^2r^2s^2\\
&\quad -p^2s^4-q^4r^2+q^4s^2+q^2r^4+2q^2r^2s^2+q^2s^4)v^2\}^2/k_2^2,
\end{aligned} \tag{3.29}$$

where $k_2=(p^2+q^2)(r^2+s^2)(u^2+v^2)$. We omit giving the remaining entries of the matrix B explicitly as these are cumbersome to write and, in any case, are not required for subsequent computations.

Finally, in the semi-magic matrix A, we replace u by $(2pq+2rs)u+(p^2-q^2-r^2+s^2)v$ and v by $(p^2-q^2-r^2+s^2)u-(2pq+2rs)v$ to get a third semi-magic matrix $C=(c_{ij})$ whose entries $c_{11},c_{12},c_{13},c_{23},c_{32}$ and c_{33} are given by

$$\begin{aligned}
c_{1j} &= a_{1j},\quad j=1,2,3,\quad c_{23}=a_{23},\quad c_{33}=a_{33},\\
c_{32} &= 4\{(p^5q+3p^4rs+2p^3q^3+2p^3qr^2-2p^3qs^2-2p^2q^2rs\\
&\qquad -2p^2r^3s+2p^2rs^3+pq^5-2pq^3r^2+2pq^3s^2-3pqr^4\\
&\qquad +2pqr^2s^2-3pqs^4+3q^4rs+2q^2r^3s-2q^2rs^3-r^5s
\end{aligned}$$

$$
\begin{aligned}
&- 2r^3s^3 - rs^5)u^2 + (p^6 + p^4q^2 - p^4r^2 + p^4s^2\\
&- 8p^3qrs - p^2q^4 - 10p^2q^2r^2 + 10p^2q^2s^2 - p^2r^4\\
&- 10p^2r^2s^2 - p^2s^4 + 8pq^3rs - 8pqr^3s + 8pqrs^3 - q^6\\
&- q^4r^2 + q^4s^2 + q^2r^4 + 10q^2r^2s^2 + q^2s^4 + r^6 + r^4s^2\\
&- r^2s^4 - s^6)uv + (-p^5q - 3p^4rs - 2p^3q^3 - 2p^3qr^2\\
&+ 2p^3qs^2 + 2p^2q^2rs + 2p^2r^3s - 2p^2rs^3 - pq^5\\
&+ 2pq^3r^2 - 2pq^3s^2 + 3pqr^4 - 2pqr^2s^2 + 3pqs^4 - 3q^4rs\\
&- 2q^2r^3s + 2q^2rs^3 + r^5s + 2r^3s^3 + rs^5)v^2\}^2/k_3^2,
\end{aligned}
\tag{3.30}
$$

where

$$
\begin{aligned}
k_3 = {} & (p^2 + 2pr + q^2 - 2qs + r^2 + s^2)\\
& \times (p^2 - 2pr + q^2 + 2qs + r^2 + s^2)(u^2 + v^2).
\end{aligned}
\tag{3.31}
$$

As before, we omit the values of the remaining entries of the matrix C as these are not needed.

The semi-magic matrices A, B, C satisfy all the conditions of Lemma 1 except (2.4). This last condition will also be satisfied if we choose p, q, r, s, u, v such that $b_{22} = c_{32}$. Now $b_{22} - c_{32}$ can be factored and equating one of the factors to 0, we reduce the last condition to the following quadratic equation in u, v:

$$
\begin{aligned}
&(p^5r + p^5s - p^4qr + p^4qs + 2p^3q^2r + 2p^3q^2s - 4p^3r^2s + 4p^3rs^2\\
&\quad - 2p^2q^3r + 2p^2q^3s - 4p^2qr^3 - 4p^2qs^3 + pq^4r + pq^4s - 4pq^2r^3\\
&\quad + 4pq^2s^3 - pr^5 - pr^4s - 2pr^3s^2 - 2pr^2s^3 - prs^4 - ps^5 - q^5r\\
&\quad + q^5s + 4q^3r^2s + 4q^3rs^2 + qr^5 - qr^4s + 2qr^3s^2 - 2qr^2s^3 + qrs^4\\
&\quad - qs^5)u^2 + (-2p^5r + 2p^5s - 2p^4qr - 2p^4qs - 4p^3q^2r + 4p^3q^2s\\
&\quad - 8p^3r^2s - 8p^3rs^2 - 4p^2q^3r - 4p^2q^3s - 8p^2qr^3 + 8p^2qs^3\\
&\quad - 2pq^4r + 2pq^4s + 8pq^2r^3 + 8pq^2s^3 + 2pr^5 - 2pr^4s + 4pr^3s^2\\
&\quad - 4pr^2s^3 + 2prs^4 - 2ps^5 - 2q^5r - 2q^5s - 8q^3r^2s + 8q^3rs^2\\
&\quad + 2qr^5 + 2qr^4s + 4qr^3s^2 + 4qr^2s^3 + 2qrs^4 + 2qs^5)uv\\
&\quad + (-p^5r - p^5s + p^4qr - p^4qs - 2p^3q^2r - 2p^3q^2s + 4p^3r^2s\\
&\quad - 4p^3rs^2 + 2p^2q^3r - 2p^2q^3s + 4p^2qr^3 + 4p^2qs^3 - pq^4r\\
&\quad - pq^4s + 4pq^2r^3 - 4pq^2s^3 + pr^5 + pr^4s + 2pr^3s^2 + 2pr^2s^3
\end{aligned}
$$

$$+ prs^4 + ps^5 + q^5r - q^5s - 4q^3r^2s - 4q^3rs^2 - qr^5 + qr^4s$$
$$- 2qr^3s^2 + 2qr^2s^3 - qrs^4 + qs^5)v^2 = 0. \tag{3.32}$$

This equation will have a rational solution for u, v if its discriminant, that is,

$$8(p^2 + q^2)(r^2 + s^2)\{(p - r)^2 + (q + s)^2\}\{(p + r)^2 + (q - s)^2\}$$
$$\times \{(p - s)^2 + (q - r)^2\}\{(p + s)^2 + (q + r)^2\} \tag{3.33}$$

is a perfect square. Trials were performed over the range $|p| + |q| + |r| + |s| < 560$ to find integer values of p, q, r, s such that the discriminant (3.33) is a perfect square, and then sextuples of integers whose pairwise sums are squares were obtained by applying Lemma 1. This yielded five sextuples of distinct integers with the values of (p, q, r, s) being given by $(1, 8, 41, 1)$, $(4, 44, 14, 5)$, $(50, 81, 124, 52)$, $(1, 73, 59, 84)$ and $(7, 119, 88, 177)$ while the corresponding five sextuples, listed in order, are as follows:

(i) 339323777731946898, 1393697157060854002, 2146648434867118098,
8397374854916636127, 12982930841197954098, −303704776155745998;

(ii) 1496507755392288, 1564461176256288, 2775230995539312,
5786742381249312, 10838537523043713, −1007706949246688;

(iii) 500754918310792773856337, 1060506697144398466471712,
2417810138225174493064688, 2432095342156176508390688,
20512561681127752879130912, −493294767960376683865312;

(iv) 1139596306822578038991888482, 1623354367647450861879856418,
7105202950963919247237173543, 21829288821309899593264932482,
45399611110517282013976271618, −538349357408323501940711518;

(v) 3721189403767988073408102698882, 4646205015271310737091409998882,
11055476621523984933834345806018, 13971965269891138856577025425143,
50909665530026992140122261302082, −3044738684647230691870004229982.

The frequency of solutions suggests the possibility of either a parametric solution of the problem or the existence of infinitely many integer solutions associated with an elliptic curve of positive rank. Efforts to find infinitely many sextuples of distinct integers were however futile.

It remains an open problem whether there exist infinitely many examples of sextuples of distinct integers such that the sum of any two of them is a perfect square. Further all the known examples of sextuples of distinct integers whose pairwise sums are squares contain one negative integer. It would be of interest to find a sextuple consisting of distinct positive integers such that the sum of any two of them is a perfect square.

References

[1] A. Choudhry, Quintuples of positive integers whose sums in pairs or in triples are squares, *Enseign. Math.* **59** (2013) 337–349.
[2] J. Cremona, Elliptic curves/Q; http://www.lmfdb.org/EllipticCurve/Q/.
[3] L. E. Dickson, *History of Theory of Numbers*, Vol. 2 (Chelsea Publishing Company, New York, 1992).
[4] R. K. Guy, *Unsolved Problems in Number Theory*, 3rd edn. (Springer, New York, 2004).
[5] J. Lagrange, Cinq nombres dont les sommes deux à deux sont des carrés, *Sém. Delange-Pisot-Poitou. Théor. Nombres* **12**(20) (1970–71) 1–10.
[6] ———, Six entiers dont les sommes deux à deux sont des carrés, *Acta Arith.* **40** (1981) 91–96.
[7] A. R. Thatcher, Five integers whose sums in pairs are squares, *Math. Gaz.* **62** (1978) 25–29.