

Area magic squares of order 3

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1 Introduction

A magic square of order 3 consists of 9 numbers $A_{ij} \in \mathbb{N}$ (usually), with $i, j \in \{1, 2, 3\}$, such that the sum of the elements in each row, column and diagonal equals the same number S .

| | | |
|----------|----------|----------|
| A_{11} | A_{12} | A_{13} |
| A_{21} | A_{22} | A_{23} |
| A_{31} | A_{32} | A_{33} |

1.1: Definition of A_{ij}

| | | |
|-----------|-----------|-----------|
| $c-b$ | $c+(a+b)$ | $c-a$ |
| $c-(a-b)$ | c | $c+(a-b)$ |
| $c+a$ | $c-(a+b)$ | $c+b$ |

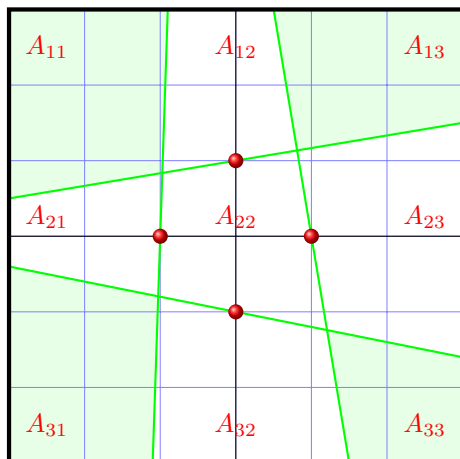
1.2: Pattern, $S=3 \cdot c$

| | | |
|----|-----|-----|
| 41 | 113 | 59 |
| 89 | 71 | 53 |
| 83 | 29 | 101 |

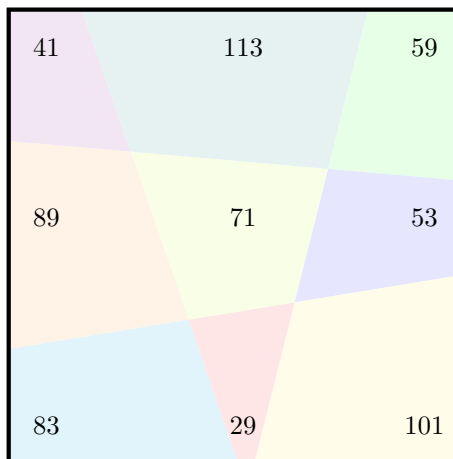
1.3: Example with $A_{ij} \in \mathbb{P}$, $S=213$

Finding a magic square is not that difficult especially if one uses the pattern given above (1.2). If we search for prime numbers, i.e. $A_{ij} \in \mathbb{P}$, we necessarily have that $a, b \equiv 0 \pmod{6}$. In the example (1.3) we have $a=12$ and $b=30$, both of which are divisible by 6.

The A_{ij} in figure 1.3 bear no relation to the area of the square in which they are depicted. The aim of this paper is to find a subdivision of the square into 9 regions by using 4 straight dissecting lines, such that each area of a region matches the corresponding value of A_{ij} , given that these numbers constitute a magic square.



1.4: Dissecting lines, fixed points



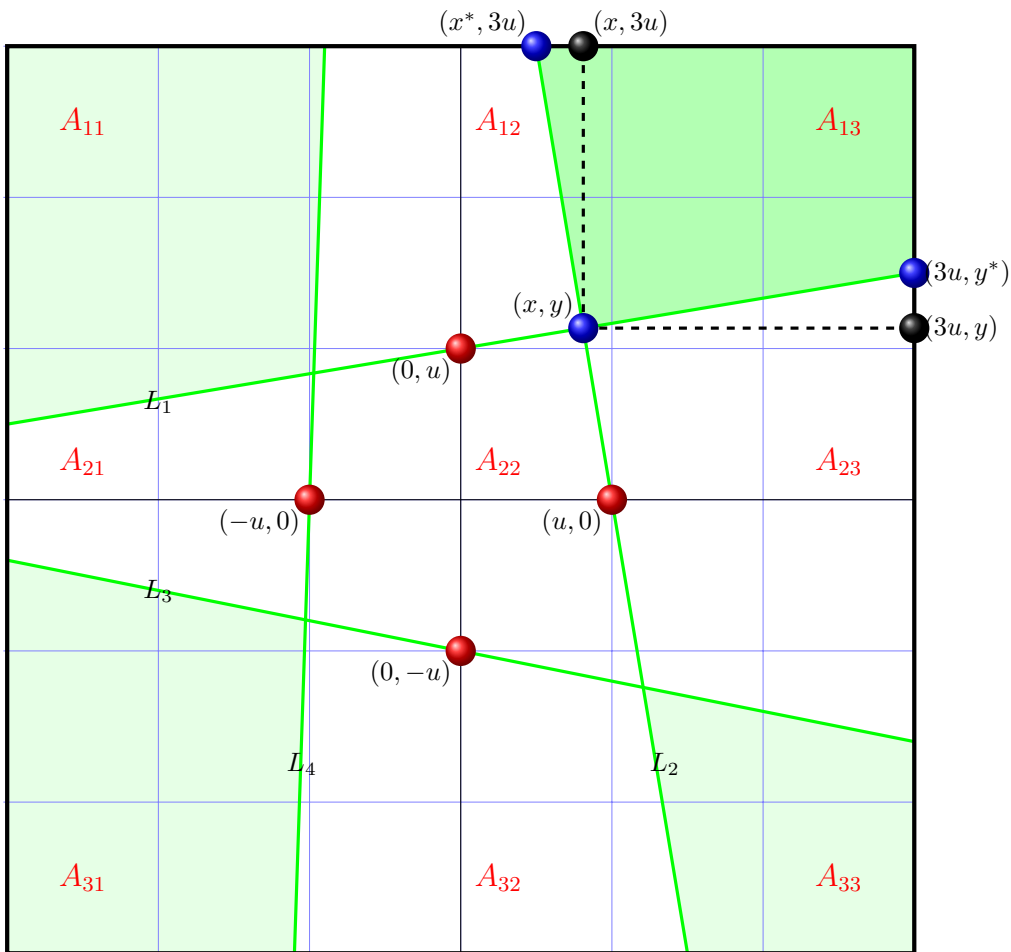
1.5: Area magic prime square

In a blog maintained by William Walkington ¹ it is shown by Walter Trump that these straight lines have fixed points (shown as red dots in figure 1.4). He found solutions to this problem using the angle of these lines as a parameter. In the same blog the results of Francis Gaspalou are found. For each of the lines he used the slope as a parameter and found equations relating $A_{11}, A_{13}, A_{31}, A_{33}$ to the 4 slopes belonging to the dissecting lines (shown as green lines in figure 1.4) and S .

2 Model

Modelling the relation between the areas $A_{11}, A_{13}, A_{31}, A_{33}$ and the parametrisation of the dissecting lines will follow the method employed by Francis Gaspalou, with a few changes:

- The square $[-3u, 3u] \times [-3u, 3u]$ is used, i.e. the origin is in the center of the square. This might facilitate simplification of the expressions involved.
- The inverse of the slope is used as a parameter for the two near vertical lines L_2 and L_4 . For the other two lines, L_1 and L_3 , the slope is used. This might keep the computation of these parameters numerically stable, especially if $A_{11} \approx A_{31}$ or $A_{13} \approx A_{33}$.



2.1: Model layout

Since the total area of this square is $(6u)^2$ which should equal $3S$ we have $12u^2 = S$.

2.1 Equations of dissection lines

$$L_1: \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ u \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ m_1 \end{pmatrix}, \quad L_2: \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix} + \beta \begin{pmatrix} m_2 \\ 1 \end{pmatrix}, \quad L_3: \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -u \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ m_3 \end{pmatrix}, \quad L_4: \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -u \\ 0 \end{pmatrix} + \delta \begin{pmatrix} m_4 \\ 1 \end{pmatrix}$$

¹<https://carresmagiques.blogspot.nl/2017/01/area-magic-squares-and-tori-of-order-3.html>

2.2 Intersection and boundarypoints

$$\begin{aligned} \text{Intersection } L_1 \cap L_2 : \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{u}{(1 - m_1 m_2)} \begin{pmatrix} 1 + m_2 \\ 1 + m_1 \end{pmatrix} \text{ and boundary } L_1, L_2 : \begin{pmatrix} x^* \\ y^* \end{pmatrix} = u \begin{pmatrix} 1 + 3m_2 \\ 1 + 3m_1 \end{pmatrix} \\ \text{Intersection } L_2 \cap L_3 : \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{u}{(1 - m_2 m_3)} \begin{pmatrix} 1 - m_2 \\ -1 + m_3 \end{pmatrix} \text{ and boundary } L_2, L_3 : \begin{pmatrix} x^* \\ y^* \end{pmatrix} = u \begin{pmatrix} 1 - 3m_2 \\ -1 + 3m_3 \end{pmatrix} \\ \text{Intersection } L_3 \cap L_4 : \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{u}{(1 - m_3 m_4)} \begin{pmatrix} -1 - m_4 \\ -1 - m_3 \end{pmatrix} \text{ and boundary } L_3, L_4 : \begin{pmatrix} x^* \\ y^* \end{pmatrix} = u \begin{pmatrix} -1 - 3m_4 \\ -1 - 3m_3 \end{pmatrix} \\ \text{Intersection } L_4 \cap L_1 : \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{u}{(1 - m_4 m_1)} \begin{pmatrix} -1 + m_4 \\ 1 - m_1 \end{pmatrix} \text{ and boundary } L_4, L_1 : \begin{pmatrix} x^* \\ y^* \end{pmatrix} = u \begin{pmatrix} -1 + 3m_4 \\ 1 - 3m_1 \end{pmatrix} \end{aligned}$$

2.3 Conditions on the parameters

The lines L_1, L_3 should intersect outside or on the boundary of the main square. The quadrilateral around either A_{21} or A_{23} degenerates into a triangle if the intersection is on the boundary. This will give a condition on m_1, m_3 . A similar condition can be derived by comparing the x^* values belonging to L_2, L_4 .

$$|m_1 - m_3| \leq 2/3 \text{ and } |m_2 - m_4| \leq 2/3$$

2.4 Calculation of the area

$$A_{13} = (3u - x)(3u - y) + ((x - x^*)(3u - y) + (3u - x)(y - y^*))/2$$

Substitute the values for x, y, x^*, y^* , set $u^2 = S/12$ and multiply by $24(1 - m_1 m_2)^2$:

$$24(1 - m_1 m_2)^2 A_{13} = S(2(3m_1 m_2 + m_1 - 2)(3m_1 m_2 + m_2 - 2) - m_1(3m_1 m_2 + m_2 - 2)^2 - m_2(3m_1 m_2 + m_1 - 2)^2)$$

This equation can be written as an expression in $m_1 m_2$ and $m_1 + m_2$:

$$24(1 - m_1 m_2)^2 A_{13} = S(- (m_1 + m_2)(9(m_1 m_2)^2 - 17(m_1 m_2) + 8) + 6(m_1 m_2)^2 - 14(m_1 m_2) + 8)$$

This can be further simplified to:

$$\begin{aligned} 24(1 - m_1 m_2) A_{13} &= S((m_1 + m_2)(9m_1 m_2 - 8) + 8 - 6m_1 m_2) \text{ or} \\ 24(1 - m_1 m_2) A_{13} &= S((m_1 + m_2)(1 - 9(1 - m_1 m_2)) + 2 + 6(1 - m_1 m_2)) \end{aligned}$$

This shows that $(1 - m_1 m_2)$ is not a factor of the right hand side of the last equation.

2.5 Equations relating area, parameters and magic sum

$$\begin{aligned} 24(1 - m_4 m_1)^2 A_{11} &= S(2(3m_4 m_1 - m_4 - 2)(3m_4 m_1 - m_1 - 2) + m_4(3m_4 m_1 - m_1 - 2)^2 + m_1(3m_4 m_1 - m_4 - 2)^2) \\ 24(1 - m_2 m_3)^2 A_{33} &= S(2(3m_2 m_3 - m_2 - 2)(3m_2 m_3 - m_3 - 2) + m_2(3m_2 m_3 - m_3 - 2)^2 + m_3(3m_2 m_3 - m_2 - 2)^2) \\ 24(1 - m_1 m_2)^2 A_{13} &= S(2(3m_1 m_2 + m_1 - 2)(3m_1 m_2 + m_2 - 2) - m_1(3m_1 m_2 + m_2 - 2)^2 - m_2(3m_1 m_2 + m_1 - 2)^2) \\ 24(1 - m_3 m_4)^2 A_{31} &= S(2(3m_3 m_4 + m_3 - 2)(3m_3 m_4 + m_4 - 2) - m_3(3m_3 m_4 + m_4 - 2)^2 - m_4(3m_3 m_4 + m_3 - 2)^2) \end{aligned}$$

These equations can be written as expressions in $m_i m_j$ and $m_i + m_j$:

$$\begin{aligned} 24(1 - m_4 m_1)^2 A_{11} &= S((m_4 + m_1)(9(m_4 m_1)^2 - 17(m_4 m_1) + 8) + 6(m_4 m_1)^2 - 14(m_4 m_1) + 8) \\ 24(1 - m_2 m_3)^2 A_{33} &= S((m_2 + m_3)(9(m_2 m_3)^2 - 17(m_2 m_3) + 8) + 6(m_2 m_3)^2 - 14(m_2 m_3) + 8) \\ 24(1 - m_1 m_2)^2 A_{13} &= S(- (m_1 + m_2)(9(m_1 m_2)^2 - 17(m_1 m_2) + 8) + 6(m_1 m_2)^2 - 14(m_1 m_2) + 8) \\ 24(1 - m_3 m_4)^2 A_{31} &= S(- (m_3 + m_4)(9(m_3 m_4)^2 - 17(m_3 m_4) + 8) + 6(m_3 m_4)^2 - 14(m_3 m_4) + 8) \end{aligned}$$

This can be further simplified to (2.5.1):

$$\begin{aligned} 24(1 - m_4 m_1) A_{11} &= S(- (m_4 + m_1)(9m_4 m_1 - 8) + 8 - 6m_4 m_1) \\ 24(1 - m_2 m_3) A_{33} &= S(- (m_2 + m_3)(9m_2 m_3 - 8) + 8 - 6m_2 m_3) \\ 24(1 - m_1 m_2) A_{13} &= S((m_1 + m_2)(9m_1 m_2 - 8) + 8 - 6m_1 m_2) \\ 24(1 - m_3 m_4) A_{31} &= S((m_3 + m_4)(9m_3 m_4 - 8) + 8 - 6m_3 m_4) \end{aligned}$$

2.6 Invariance relation between the parameters

Using $24(A_{11} + A_{33}) = 16S$ and $24(A_{13} + A_{31}) = 16S$ one gets 2 equations relating the m_i . If one adds these we get $24(A_{11} + A_{33} + A_{13} + A_{31}) = 32S$, which simplifies the resulting equation considerably, due to cancellation.

$$9(m_1 - m_3)(m_2 - m_4)(m_1 + m_2 + m_3 + m_4) + 2(m_1m_2 + m_2m_3 + m_3m_4 + m_4m_1) = 8m_1m_2m_3m_4 \quad (2.6.1)$$

$$\text{If the } A_{ij} \text{ are approximately equal we should have: } m_1m_2 + m_2m_3 + m_3m_4 + m_4m_1 \approx 0. \quad (2.6.2)$$

3 Algorithm

The 4 equations (2.5.3) can be written in the form $F_i(m_1, m_2, m_3, m_4) = 0$, each is a polynomial in two of the m_i and of order 2. It is advised to divide both sides of (2.5.1) by S .

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Initialise:  $\mathbf{m} := \mathbf{m}_0$ 

repeat ( $i \geq 0$ ):
  solve  $\mathbf{F}(\mathbf{m}_i) + \mathbf{J}(\mathbf{m}_i) \cdot d\mathbf{m}_i = \mathbf{0}$ 
  update  $\mathbf{m}_{i+1} := \mathbf{m}_i + d\mathbf{m}_i$ 
until  $\|d\mathbf{m}_i\|$  is small enough

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Boldface characters ($\mathbf{F}, \mathbf{m}, d\mathbf{m}, \mathbf{0}$) are chosen to indicate that vectors are implied. \mathbf{J} equals the Jacobian of \mathbf{F} . In a sense this is just an extension to the Newton-Raphson method. We have $\mathbf{m}_{i+1} = \mathbf{m}_i - \mathbf{J}(\mathbf{m}_i)^{-1} \cdot \mathbf{F}(\mathbf{m}_i)$. Instead of computing the inverse of $\mathbf{J}(\mathbf{m}_i)$ it is recommended to use a **LU**-decomposition. This algorithm might behave badly, because the initial guess \mathbf{m}_0 must be close enough to the final solution in order for this procedure to work, especially if \mathbf{J} is nearly singular. Note that $\mathbf{J}(\mathbf{0})$ is singular.

3.1 Initial values

In order to estimate initial values for the m_i we could imagine each area A_{ij} to be square.

$$\widehat{m}_1 \approx \frac{\sqrt{A_{11}} - \sqrt{A_{13}}}{\sqrt{S - A_{11} - A_{13}}}, \widehat{m}_2 \approx \frac{\sqrt{A_{33}} - \sqrt{A_{13}}}{\sqrt{S - A_{33} - A_{13}}}, \widehat{m}_3 \approx \frac{\sqrt{A_{33}} - \sqrt{A_{31}}}{\sqrt{S - A_{33} - A_{31}}}, \widehat{m}_4 \approx \frac{\sqrt{A_{11}} - \sqrt{A_{31}}}{\sqrt{S - A_{11} - A_{31}}} \quad (3.1.1)$$

These can be simplified by writing $3A_{11} = S - \delta$, $3A_{33} = S + \delta$, $3A_{13} = S - \epsilon$ and $3A_{31} = S + \epsilon$. A zero order approximation in terms of δ and ϵ gives:

$$\widehat{m}_1 \approx \frac{3}{2} \frac{A_{11} - A_{13}}{S}, \widehat{m}_2 \approx \frac{3}{2} \frac{A_{33} - A_{13}}{S}, \widehat{m}_3 \approx \frac{3}{2} \frac{A_{33} - A_{31}}{S}, \widehat{m}_4 \approx \frac{3}{2} \frac{A_{11} - A_{31}}{S} \quad (3.1.2)$$

These estimators have the property that $\widehat{m}_1 = -\widehat{m}_3$ and $\widehat{m}_2 = -\widehat{m}_4$. Substituting into relation (2.6.1) reveals that this approximation is good if $\widehat{m}_1\widehat{m}_2\widehat{m}_3\widehat{m}_4$ is small, i.e. if the A_{ij} are approximately equal. Relation (2.6.2) is satisfied by definition. These estimators give the approximations:

$$\frac{A_{11}}{A_{33}} \approx \frac{1 + \widehat{m}_1 - \widehat{m}_2}{1 + \widehat{m}_3 - \widehat{m}_4}, \frac{A_{13}}{A_{31}} \approx \frac{1 - \widehat{m}_1 - \widehat{m}_2}{1 - \widehat{m}_3 - \widehat{m}_4}$$

3.2 Example

In figure (1.5) the following values are used: $A_{11} = 41, A_{13} = 59, A_{31} = 83, A_{33} = 101$. We will use (3.1.1) since the A_{ij} are not approximately equal. The Euclidean norm is used.

$$\begin{aligned} \mathbf{m}_0 &= (-0.1202, 0.3254, 0.1745, -0.2870)^T, \mathbf{F}_0 = (82.3471, -142.3556, 121.2045, -42.9193)^T, \\ d\mathbf{m}_0 &= (0.1780, -0.1998, 0.1105, -0.1960)^T, \|d\mathbf{m}_0\| = 0.3496 \\ \mathbf{m}_1 &= (0.0578, 0.1255, 0.2849, -0.4830)^T, \mathbf{F}_1 = (20.8373, 7.7428, -3.5733, 19.1383)^T, \\ d\mathbf{m}_1 &= (-0.0928, 0.0784, -0.0765, 0.0889)^T, \|d\mathbf{m}_1\| = 0.1689 \\ \mathbf{m}_2 &= (-0.0351, 0.2040, 0.2084, -0.3941)^T, \mathbf{F}_2 = (3.1672, 2.1808, 0.9552, 3.9873)^T, \\ d\mathbf{m}_2 &= (-0.0390, 0.0347, -0.0335, 0.0370)^T, \|d\mathbf{m}_2\| = 0.0370 \\ \mathbf{m}_3 &= (-0.0741, 0.2387, 0.1749, -0.3571)^T, \mathbf{F}_3 = (0.5385, 0.4110, 0.3107, 0.7126)^T, \\ d\mathbf{m}_3 &= (-0.0120, 0.0108, -0.0104, 0.0113)^T, \|d\mathbf{m}_3\| = 0.0223 \end{aligned}$$

$$\begin{aligned} \mathbf{m}_4 &= (-0.0861, 0.2495, 0.1646, -0.3458)^T, \mathbf{F}_4 = (0.0494, 0.0389, 0.0326, 0.0683)^T, \\ \mathbf{dm}_4 &= (-0.0014, 0.0013, -0.0012, 0.0013)^T, \|\mathbf{dm}_4\| = 0.0026 \\ \mathbf{m}_5 &= (-0.0875, 0.2507, 0.1633, -0.3445)^T, \mathbf{F}_5 = (0.0007, 0.0005, 0.0005, 0.0010)^T, \\ \mathbf{dm}_5 &= (-0.00002, 0.00002, -0.00002, 0.00002)^T, \|\mathbf{dm}_5\| = 0.00004 \end{aligned}$$

The algorithm behaves poorly, especially at the start. Initially we have $\|\mathbf{m}_0\| = 0.4829$ and the adaptation $\|\mathbf{dm}_0\| = 0.3496$ which is 72% of the initial norm. Either we need to have much better initial estimates or try to improve on the algorithm itself.

4 Improved algorithm

Before entering the loop we must make sure that the first step doesn't overshoot the final solution. In the hope that the general direction of the adaption \mathbf{dm}_0 is correct one could try to downsize the stepsize by choosing a small α , say 0.10, and compute:

$$\mathbf{dm}_{0*} := \frac{\mathbf{dm}_0}{\|\mathbf{dm}_0\|} \alpha \|\mathbf{m}\| \text{ as a consequence } \|\mathbf{dm}_{0*}\| = \alpha \|\mathbf{m}\|$$

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Initialise:  $\mathbf{m} := \mathbf{m}_0$ 

solve  $\mathbf{F}(\mathbf{m}_0) + \mathbf{J}(\mathbf{m}_0) \cdot \mathbf{dm}_0 = \mathbf{0}$ 
norm  $\mathbf{dm}_0 := \frac{\mathbf{dm}_0}{\|\mathbf{dm}_0\|} \alpha \|\mathbf{m}\|$ 
update  $\mathbf{m}_1 := \mathbf{m}_0 + \mathbf{dm}_0$ 

repeat ( $i > 0$ ):
  solve  $\mathbf{F}(\mathbf{m}_i) + \mathbf{J}(\mathbf{m}_i) \cdot \mathbf{dm}_i = \mathbf{0}$ 
  update  $\mathbf{m}_{i+1} := \mathbf{m}_i + \mathbf{dm}_i$ 
until  $\|\mathbf{dm}_i\|$  is small enough

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4.1 Example revisited

In figure (1.5) the following values are used: $A_{11} = 41, A_{13} = 59, A_{31} = 83, A_{33} = 101$. We will use (3.1.1) since the A_{ij} are not approximately equal. We'll take $\alpha = 0.1$.

$$\begin{aligned} \mathbf{m}_0 &= (-0.1202, 0.3254, 0.1745, -0.2870)^T, \mathbf{F}_0 = (82.3471, -142.3556, 121.2045, -42.9193)^T, \\ \mathbf{dm}_0 &= (0.1780, -0.1998, 0.1105, -0.1960)^T, \|\mathbf{dm}_0\| = 0.3496, \|\mathbf{m}_0\| = 0.4828, \\ \text{multiply } \mathbf{dm}_0 &\text{ by } 0.1 \cdot 0.4828/0.3496 = 0.1381, \mathbf{dm}_0 = (0.0246, -0.0276, 0.0152, -0.0270)^T \\ \mathbf{m}_1 &= (-0.0956, 0.2978, 0.1897, -0.3140)^T, \mathbf{F}_1 = (0.3352, -0.5756, 0.4904, -0.1721)^T, \\ \mathbf{dm}_1 &= (0.0730, -0.0726, -0.0023, -0.0578)^T, \|\mathbf{dm}_1\| = 0.1001 \\ \mathbf{m}_2 &= (-0.0582, 0.2252, 0.1874, -0.3718)^T, \mathbf{F}_2 = (0.0116, 0.0095, -0.0058, 0.0065)^T, \\ \mathbf{dm}_2 &= (-0.0242, 0.0210, -0.0196, 0.025)^T, \|\mathbf{dm}_2\| = 0.0438 \\ \mathbf{m}_3 &= (0.0824, 0.2461, 0.1678, -0.3493)^T, \mathbf{F}_3 = (0.0009, 0.0007, 0.0005, 0.0013)^T, \\ \mathbf{dm}_3 &= (-0.0049, 0.0044, -0.0042, 0.0046)^T, \|\mathbf{dm}_3\| = 0.0091 \\ \mathbf{m}_4 &= (-0.0873, 0.2505, 0.1635, -0.3447)^T, \mathbf{F}_4 = (0.00004, 0.00003, 0.00002, 0.00005)^T, \\ \mathbf{dm}_4 &= (-0.00024, 0.00022, -0.00021, 0.00023)^T, \|\mathbf{dm}_4\| = 0.00045 \\ \mathbf{m}_5 &= (-0.0875, 0.2508, 0.1633, -0.3444)^T, \|\mathbf{F}_5\| = 2 \cdot 10^{-7}, \|\mathbf{dm}_5\| = 1 \cdot 10^{-6}. \end{aligned}$$

4.2 Example palindromic primes

The following values are used²: $A_{11} = 10797779701, A_{13} = 12568586521, A_{31} = 12566366521, A_{33} = 14337173341$. Here $S = 37702429563$. We will use (3.1.2) and $\alpha = 0.1$ and 12 significant digits.

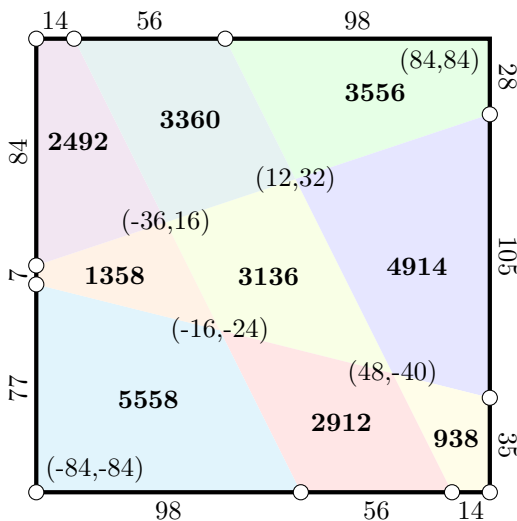
$$\begin{aligned} \mathbf{m}_5 &= (-0.0768414564889, 0.0753157136185, 0.0643670825273, -0.0653250521210)^T, \\ \|\mathbf{F}_5\| &= 4 \cdot 10^{-20}, \|\mathbf{dm}_5\| = 2 \cdot 10^{-18}. \end{aligned}$$

²Contributed by Carlos Rivera and Jaime Ayala on May 22, 1999 to <http://magic-squares.net>

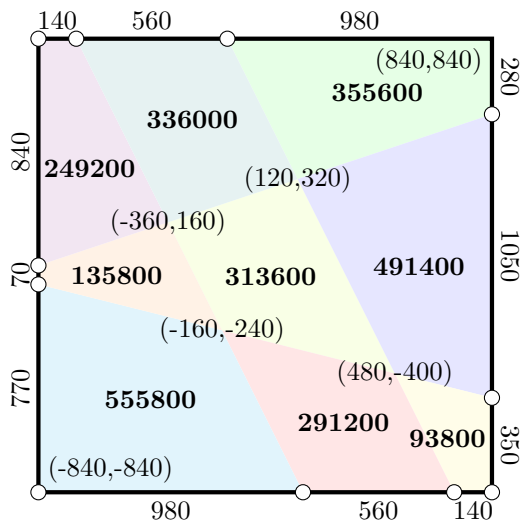
5 Area semi-magic squares with integer coordinates

It would be nice if the coordinates of the polygons are in \mathbb{N} as well. It is possible to prove that the magic sum must be even. However no solution was found with the property that the diagonal sums were equal to the magic square sum S . All squares do have the property that the sum of both diagonals is equal to $2S$, the sum of the elements in the rows and columns do add to S , hence the name area semi-magic squares.

With $S = 12u^2$ all semi-magic squares with $u \leq 300$ have the property that u has at least two factors. No solutions with u odd are found. The solutions are in the set $\{28, 56, 84, 110, 112, 120, 140, 168, 180, 182, 196, 200, 220, 224, 240, 252, 280\}$. It helps if u is a highly composite number in order to compute the various intersection points.



5.1: $S = 12 \cdot 28^2 = 9408$

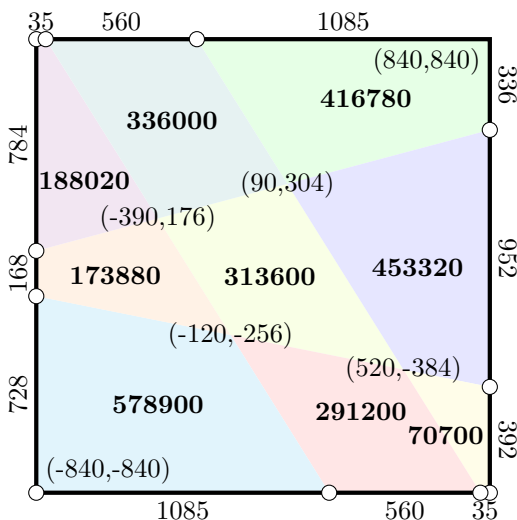


5.2: $S = 12 \cdot 280^2 = 940800$

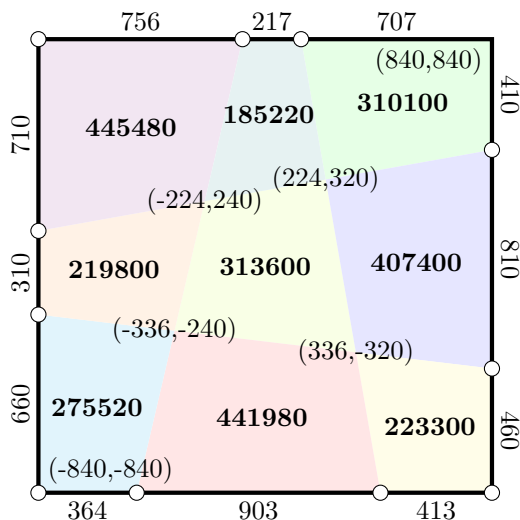
Most values of S give area semi-magic squares with parallel dissection lines as shown above. The area semi-magic squares above are congruent, however $S=9408$ has only 1 solution (modulo symmetries).

If all area semi-magic squares have at least one configuration with parallel lines, one could specify $m_2 = m_4$, choose m_1, m_2 substitute in (2.6.1) and calculate m_3 (speeding up the calculations!):

$$m_1 + m_3 = 2m_2m_1m_3$$



5.3: $S = 12 \cdot 280^2 = 940800$



5.4: $S = 12 \cdot 280^2 = 940800$

Only $u = 280$ gives a solution where all the distances alongside the border of the square are different (5.4).

6 Shoelace formula

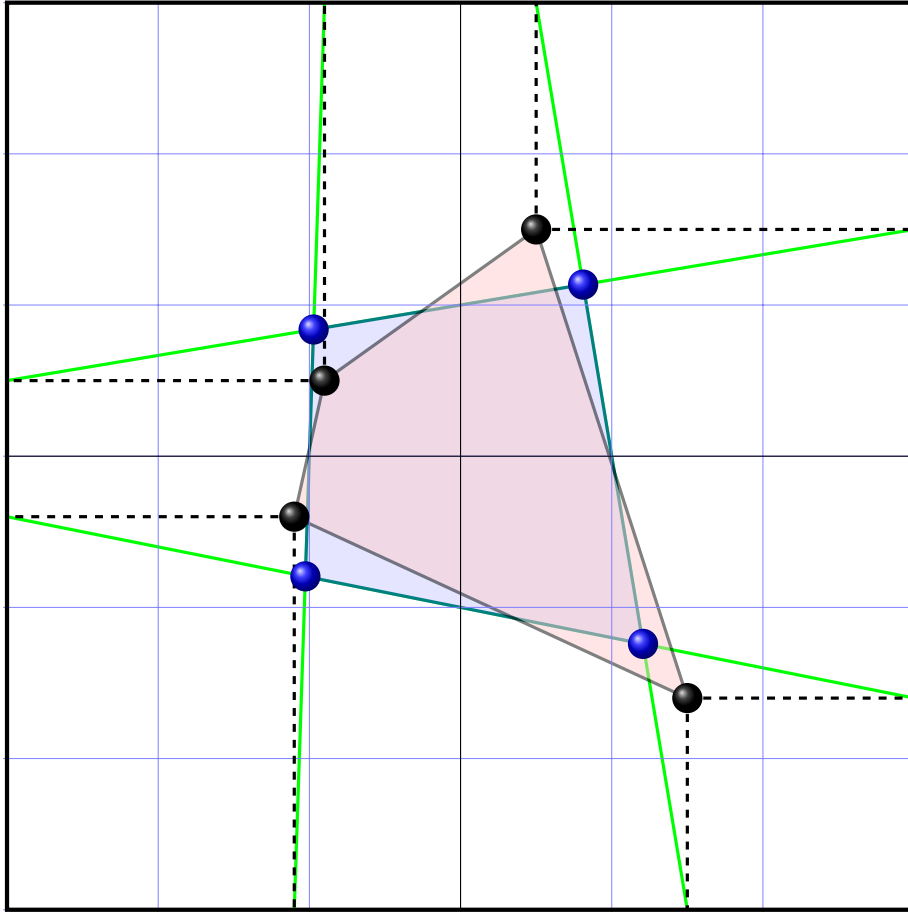
The Shoelace formula³ relates the area of a non self-intersecting-polygon to the coordinates in a Cartesian plane. Applied to 4 vertices of a polygon (visited counterclockwise, like all mathematicians tend to do 😊):

$$A = \frac{1}{2}((x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_1y_4 + x_2y_1 + x_3y_2 + x_4y_3)) \quad (6.1)$$

If this formula is applied to the 4 points defined by the coordinates (x^*, y^*) of the boundary points described in section 2.2 we get the following relation assuming $4u^2 = S/3$, a property of area magic squares:

$$A = S\left(\frac{1}{3} - \frac{3}{4}(m_1m_2 + m_2m_3 + m_3m_4 + m_4m_1)\right) \quad (6.2)$$

Which gives a geometric interpretation to part of the invariance relation (the left hand side of (2.6.2)), in a sense that it measures the deviation in area from the central quadrilateral.



5.1: Geometric interpretation of $m_1m_2 + m_2m_3 + m_3m_4 + m_4m_1$

The blue area equals $S/3$ (assuming an area magic square), the red area equals A . Upon reordering:

$$m_1m_2 + m_2m_3 + m_3m_4 + m_4m_1 = \frac{4}{9} \left(1 - \frac{A}{S/3}\right) \quad (6.3)$$

If we drop the link with area magic squares (but retain the fixed points):

$$m_1m_2 + m_2m_3 + m_3m_4 + m_4m_1 = \frac{4}{9} \left(1 - \frac{A}{T/9}\right) \quad \text{with } T \text{ the total area of the square} \quad (6.4)$$

If $m_1 = -m_3$ or $m_2 = -m_4$ we have $A = T/9$.

³https://en.wikipedia.org/wiki/Shoelace_formula