On the Puzzle 1136

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Puzzle Q1

The following theorem proves only one implication of the Q1 and shows the general form the various primes must have.

Theorem. Let p be a prime integer. If $G(p) = \sigma(p-1)$ is prime, then $F(p) = d(p-1)$ is prime too. Moreover, either $p = F_0 = 3$ or $p = F_{2^n}$, $F(p) = F_n$ and $G(p) = M_{F(p)}$, where F_N and M_N denote Fermat and Mersenne numbers respectively.

Proof. Since $\sigma(n)$ is a multiplicative function, the primality of $G(p)$ forces $n = q^\alpha$ to avoid a trivial factorization, with q prime and $\alpha > 0$. It follows that

$$p = q^\alpha + 1, \quad F(p) = \alpha + 1, \quad G(p) = \frac{q^{F(p)} - 1}{q - 1}$$

Therefore, if $G(p)$ is prime then the exponent $F(p)$ must be prime to avoid trivial divisors. Let's underline that this fact is indipendent of the primality of p, i.e. $\sigma(N)$ prime $\implies d(N)$ prime for every integer N.

If the condition p and q primes is used, then we must have $\alpha = 2^k$ and $q = 2$, hence p is Fermat prime. Besides, $F(p)$ is prime and

$$F(p) = 2^k + 1$$

so by the same reasoning either $k = 2^n$ or $k = 0$. In the latter case, $p = 3 = F_0$ and $G(p) = 3$, both prime, hence the theorem conditions are satisfied. Alternatively it follows that

$$p = F_{2^n} = 2^{F_n - 1} + 1, \quad F(p) = F_n = 2^{2^n} + 1, \quad G(p) = M_{F(p)} = 2p - 3$$

where all three are prime. □

Puzzle Q2

The first 6 terms of this form are

$$\begin{aligned} p_N &= F_0,\ F_1,\ F_2,\ F_4,\ F_8,\ F_{16} \\ F(p_N) &= 2,\ F_0,\ F_1,\ F_2,\ F_3,\ F_4 \\ G(p_N) &= M_2,\ M_3,\ M_5,\ M_{17},\ M_{257},\ M_{65537} \end{aligned}$$

that satisfy $F(p_N) = p_{N-1}$ until $N = 5$, but in that case F_8 and M_{257} are composite. Therefore the formula would still be valid if $(F_{16},\ F_4,\ M_{65537})$ were all prime, however only F_4 is actually prime. Consequentially, the formula of Q2 is true iff there are no other prime triplet. Let me point out that $p_7 = F_{32}$ is known composite.

Puzzle Q3

$257 = F_3$ doesn't appear because 3 isn't a power of 2, thus $F(p) = 2^3 + 1$ is trivially composite.

Puzzle Q4

As discussed in Q2, that would be $p = F_{16}$, however is not prime.